

OPTIMALLY CONVERGENT HDG METHOD FOR THIRD-ORDER KORTEWEG-DE VRIES TYPE EQUATIONS

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ABSTRACT. We develop and analyze a new hybridizable discontinuous Galerkin (HDG) method for solving third-order Korteweg-de Vries type equations. The approximate solutions are defined by a discrete version of a characterization of the exact solution in terms of the solutions to local problems on each element which are patched together through transmission conditions on element interfaces. We prove that the semi-discrete schemes for the linear and nonlinear KdV type equations are stable with proper choices of stabilization function in the numerical traces. Moreover, for the linear equation, we carry out error analysis and show that the approximations to the exact solution u and its derivatives $q := u_x$ and $p := u_{xx}$ have optimal convergence rates. In numerical experiments, we use implicit schemes for time discretization and the Newton-Raphson method for solving nonlinear equations, and we observe optimal convergence rates for both the linear and the nonlinear third-order equations.

1. INTRODUCTION

In this paper, we develop and analyze a new hybridizable discontinuous Galerkin (HDG) method for the following Korteweg-de Vries (KdV) type problem [36, 35, 1, 19, 4, 3]:

$$(1.1) \quad \begin{aligned} u_t + u_{xxx} + F(u)_x &= f && \text{for } x \in \Omega := (a, b), t \in (0, T], \\ u &= u_0 && \text{in } \Omega \text{ for } t = 0, \\ u &= u_D && \text{on } \partial\Omega := \{a, b\}, \\ u_x &= q_N && \text{on } \partial\Omega_N := \{b\}. \end{aligned}$$

Here $f \in L^2(\Omega)$ and $F(u) = \beta u^m$, where β is a constant and $m \geq 0$ an integer.

KdV type equations play an important role in applications, such as fluid mechanics [25, 7, 24], nonlinear optics [2, 17], acoustics [27, 31], plasma physics [6, 30, 28], and Bose-Einstein condensates [29, 20] among other fields. They also have an enormous impact on the development of nonlinear mathematical science and theoretical physics. Many modern areas were opened up as a consequence of the basic research on KdV equations. Due to their importance in applications and theoretical studies, there has been a lot of interest in developing accurate and efficient numerical methods for KdV equations. In particular, an ongoing effort on developing discontinuous Galerkin (DG) methods for KdV type equations has been made in the last decade. The first DG method, the local discontinuous Galerkin (LDG) method, for the KdV equation was introduced in 2002 by Yan and Shu in [34] and further

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studied in the linear case in [22, 32, 33, 18]. In [10], a DG method for the *KdV* equation was devised by using repeated integration by parts. Recently, several conservative DG methods [5, 9, 21] were developed for KdV type equations to preserve quantities such as the mass and the L^2 -norm of the solutions. When solving KdV equations, one can use these DG methods for spatial discretization together with explicit schemes for time-marching if the coefficient before the third-order derivative is very small. However, when such coefficient is of order one, for example, implicit time-marching methods might be the methods of choice.

Traditional DG methods, despite of their prominent features such as hp-adaptivity and local conservativity, were criticized for having larger number of degrees of freedom than continuous finite element methods when solving steady-state problems or problems that require implicit-in-time solvers. Here, we develop an HDG method which is very suitable for solving KdV equations when implicit time-marching is used. HDG methods [13, 11, 15, 14] were first introduced for diffusion problems and they provide optimal approximations to both the potential and the flux. Due to the feature that the global coupled degrees of freedom only live on element interfaces, they are significantly advantageous for solving steady-state problems or time-dependent problems that require implicit time-marching. In [8], we introduced the first family of HDG methods for stationary third-order linear equations, which allow the approximations to the exact solution u and its derivatives u_x and u_{xx} to have different polynomial degrees. We proved superconvergence properties of these methods on projection of errors and numerical traces, and numerical results indicate that the HDG method using the same polynomial degree k for all three variables is quite robust with respect to the choice of the stabilization function and provides a converging postprocessed solution with order $2k + 1$ with the least amount of degrees of freedom. This suggests that the HDG method using the same polynomial degrees for all variables is the method of choice for solving one-dimensional third-order problems. Therefore, in this paper we extend this HDG method to time-dependent third-order KdV equations.

To construct the HDG method for KdV equations, we follow the approach used in [8] for stationary third-order equations. That is, given any mesh of the domain, we show that the exact solution can be obtained by solving the equation on each element with provided boundary data that are determined by transmission conditions. Then we define HDG methods by a discrete version of this characterization, which ensures that the only globally-coupled degrees of freedom are those associated to the numerical traces on element interfaces. In [8], it was shown that HDG methods derived by providing boundary data to local problems in different ways are indeed equivalent to each other when the stabilization parameters are finite and nonzero. So here we just need consider the one that takes the numerical trace of u at both ends of the interval and the numerical trace of u_{xx} at the right end as boundary data of the local problems. Our method is different from the HDG method in [26], which was designed from “static condensation” point of view. That HDG method involves two sets of numerical traces for u_x , and there is no error analysis for the method.

Our way of devising HDG methods from the characterization of the exact solution allows us to carry out stability and optimal error analysis. We first apply an energy argument to find conditions on the stabilization function in the numerical traces under which the HDG method has a unique solution for the linear and nonlinear

third-order problems. Then by deriving four energy identities and combining them together, we prove that the method has optimal approximations to u as well as its derivatives u_x and u_{xx} for linear equations; this technique is similar to that in [33]. In implementation, implicit time-marching schemes such as BDF or DIRK methods can be used and at each time step a steady-state third-order equation is solve by the HDG method together with the Newton-Raphson method. Due to the one-dimensional setting of the KdV equations, the global matrix of the HDG method that needs to be numerically inverted at each time step is independent of the polynomial degree of the approximations, its size is only $2N + 1$, where N is the number of intervals of the mesh, and its condition number is of the order of h^{-2} , where h denotes the size of the intervals of the mesh.

The paper is organized as follows. In Section 2, we define the HDG method for the linear and nonlinear KdV type equations and state and discuss our main results. The details of all the proofs are given in Section 3. We show the numerical results in Section 4 and finally end with some concluding remarks in Section 5.

2. MAIN RESULTS

In this section, we state and discuss our main results. We begin by describing the characterizations of the exact solution that the HDG method is a discrete version of. We then introduce our HDG method for linear KdV type equations and state our stability result and optimal a priori error estimate. After that, we define the HDG method in the nonlinear case and discuss the stability result.

2.1. Characterizations of the exact solution. To display the characterizations of the exact solution we are going to work with, let us first rewrite our third-order model equation as the following first-order system:

$$(2.1a) \quad q - u_x = 0, \quad p - q_x = 0, \quad u_t + p_x + F(u)_x = f \quad \text{for } x \in \Omega, t \in (0, T],$$

with the initial and boundary conditions

$$(2.1b) \quad u = u_0 \quad \text{in } \Omega \text{ for } t = 0,$$

$$(2.1c) \quad u = u_D \quad \text{on } \partial\Omega,$$

$$(2.1d) \quad q = q_N \quad \text{on } \partial\Omega_N.$$

We partition the domain Ω as

$$\mathcal{T}_h = \{I_i := (x_{i-1}, x_i) : a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b\},$$

and introduce the set of the boundaries of its elements, $\partial\mathcal{T}_h := \{\partial I_i : i = 1, \dots, N\}$. We also set $\mathcal{E}_h := \{x_i\}_{i=0}^N$, $h_i = x_i - x_{i-1}$ and $h := \max_{i=1}^N h_i$.

We know that, when f is smooth enough, if we provide the values $\{\widehat{u}_i\}_{i=0}^N$ and $\{\widehat{p}_i\}_{i=1}^N$ and, for each $i = 1, \dots, N$, solve the local problem

$$\begin{aligned} Q - U_x &= 0, \quad P - Q_x = 0, \quad U_t + P_x + F(U)_x = f \quad \text{in } I_i, \\ U &= u_0 \quad \text{for } t = 0, \quad U(x_{i-1}^+) = \widehat{u}_{i-1}, \quad U(x_i^-) = \widehat{u}_i, \quad P(x_i^-) = \widehat{p}_i, \end{aligned}$$

then (P, Q, U) coincides with the solution (p, q, u) of our model problem (2.1) if and only if the transmission conditions

$$Q(x_i^-) = Q(x_i^+), \quad P(x_i^-) = P(x_i^+), \quad i = 1, \dots, N-1$$

and the boundary conditions

$$U = u_D \text{ on } \partial\Omega, \quad Q = q_N \text{ on } \partial\Omega_N$$

are satisfied. There are other possible characterizations of the exact solution corresponding to different choices of boundary data for the local problem; see [8]. Note that for these characterizations, the boundary data of the local problems are the unknowns of a global problem obtained from the transmission conditions and boundary conditions, and the system of equations for the global unknowns is square.

2.2. Some notation. To define the HDG method, we first introduce the finite element spaces to be used. We let the approximations $(u_h, q_h, p_h, \hat{u}_h, \hat{q}_h, \hat{p}_h)$ to $(u|_\Omega, q|_\Omega, p|_\Omega, u|_{\mathcal{E}_h}, q|_{\mathcal{E}_h}, p|_{\mathcal{E}_h})$ be in the space $W_h^k \times W_h^k \times W_h^k \times L^2(\mathcal{E}_h) \times L^2(\partial\mathcal{T}_h) \times L^2(\partial\mathcal{T}_h)$ where

$$W_h^k = \{w \in L^2(\mathcal{T}_h) : w|_{I_i} \in P_k(I_i) \quad \forall i = 1, \dots, N\}.$$

Here $P_k(I_i)$ is the space of polynomials of degree at most k on the domain I_i . For any function ζ lying in $L^2(\partial\mathcal{T}_h)$, we denote its values on $\partial I_i := \{x_{i-1}^+, x_i^-\}$ by $\zeta(x_{i-1}^+)$ (or simply ζ_{i-1}^+) and $\zeta(x_i^-)$ (or simply ζ_i^-). Note that $\zeta(x_i^+)$ is not necessarily equal to $\zeta(x_i^-)$. In contrast, for any η in the space $L^2(\mathcal{E}_h)$, its value at x_i , $\eta(x_i)$ (or simply η_i) is uniquely defined; in this case, $\eta(x_i^-)$ or $\eta(x_i^+)$ mean nothing but $\eta(x_i)$.

To obtain the HDG formulation, we use a discrete version of the characterization of the exact solution. Assuming that the values $\{\hat{u}_{hi}\}_{i=0}^N$ and $\{\hat{p}_{hi}^-\}_{i=1}^N$ are given, for each $i = 1, \dots, N$, we solve a local problem on the element I_i by using a Galerkin method. To describe it, let us introduce the following notation. By $(\varphi, v)_{I_i}$, we denote the integral of φ times v on the interval I_i , and by $\langle \varphi, vn \rangle_{\partial I_i}$ we simply mean the expression $\varphi(x_i^-)v(x_i^-)n(x_i^-) + \varphi(x_{i-1}^+)v(x_{i-1}^+)n(x_{i-1}^+)$. Here n denotes the outward unit normal to I_i : $n(x_{i-1}^+) := -1$ and $n(x_i^-) := 1$.

2.3. HDG methods for the linear problem. We first consider (2.1) with $\beta = 0$ for $F(u) = \beta u^m$. On the element I_i , we give f and the boundary data $\hat{u}_{hi-1}, \hat{u}_{hi}$ and \hat{p}_{hi}^- and take $(p_h, q_h, u_h) \in P_k(I_i) \times P_k(I_i) \times P_k(I_i)$ to be the solution of the equations

$$\begin{aligned} (q_h, v)_{I_i} + (u_h, v_x)_{I_i} - \langle \hat{u}_h, vn \rangle_{\partial I_i} &= 0, \\ (p_h, z)_{I_i} + (q_h, z_x)_{I_i} - \langle \hat{q}_h, zn \rangle_{\partial I_i} &= 0, \\ (u_{ht}, w)_{I_i} - (p_h, w_x)_{I_i} + \langle \hat{p}_h, wn \rangle_{\partial I_i} &= (f, w)_{I_i}, \end{aligned}$$

for all $(v, z, w) \in P_k(I_i) \times P_k(I_i) \times P_k(I_i)$, where the remaining undefined numerical traces are given by

$$\begin{cases} \hat{p}_h = p_h + \tau_{pu} (\hat{u}_{hi-1} - u_h) n & \text{at } x_{i-1}^+, \\ \hat{q}_h = q_h + \tau_{qu} (\hat{u}_{hi-1} - u_h) n & \text{at } x_{i-1}^+, \\ \hat{q}_h = q_h + \tau_{qu} (\hat{u}_{hi} - u_h) n + \tau_{qp} (\hat{p}_{hi}^- - p_h) n & \text{at } x_i^-. \end{cases}$$

The functions τ_{qu}, τ_{qp} , and τ_{pu} are defined on $\partial\mathcal{T}_h$ and are called the components of the *stabilization function*; they have to be properly chosen to ensure that the above problem has a unique solution.

It remains to impose the transmission conditions

$$[\![\hat{q}_h]\!](x_i) = 0 \quad \text{and} \quad [\![\hat{p}_h]\!](x_i) = 0 \quad \text{for all } i = 1, \dots, N-1,$$

and the boundary conditions

$$\hat{u}_h = u_D \text{ on } \partial\Omega \quad \text{and} \quad \hat{q}_h = q_N \text{ on } \partial\Omega_N.$$

This completes the definition of the HDG methods using the characterization of the exact solution. Note that this way of defining the HDG methods immediately provides a way to implement them.

On the other hand, the above presentation of the HDG methods is not very well suited for their analysis. Thus, we now rewrite it in a more compact form using the notation

$$(\varphi, v)_{\mathcal{T}_h} := \sum_{i=1}^N (\phi, v)_{I_i}, \quad \langle \varphi, vn \rangle_{\partial\mathcal{T}_h} := \sum_{i=1}^N \langle \varphi, vn \rangle_{\partial I_i}.$$

The approximation provided by the HDG methods, $(u_h, q_h, p_h, \hat{u}_h, \hat{q}_h, \hat{p}_h)$, is the element of $W_h^k \times W_h^k \times W_h^k \times L^2(\mathcal{E}_h) \times L^2(\partial\mathcal{T}_h) \times L^2(\partial\mathcal{T}_h)$ which solves the equations

$$(2.2a) \quad (q_h, v)_{\mathcal{T}_h} + (u_h, v_x)_{\mathcal{T}_h} - \langle \hat{u}_h, vn \rangle_{\partial\mathcal{T}_h} = 0,$$

$$(2.2b) \quad (p_h, z)_{\mathcal{T}_h} + (q_h, z_x)_{\mathcal{T}_h} - \langle \hat{q}_h, zn \rangle_{\partial\mathcal{T}_h} = 0,$$

$$(2.2c) \quad (u_{ht}, w)_{\mathcal{T}_h} - (p_h, w_x)_{\mathcal{T}_h} + \langle \hat{p}_h, wn \rangle_{\partial\mathcal{T}_h} = (f, w)_{\mathcal{T}_h}$$

for all $(v, z, w) \in W_h^k \times W_h^k \times W_h^k$, where, on $\partial\mathcal{T}_h$, we have

$$(2.2d) \quad \begin{cases} \hat{p}_h^+ = p_h^+ + \tau_{pu}^+ (\hat{u}_h - u_h^+) n^+, \\ \hat{q}_h^+ = q_h^+ + \tau_{qu}^+ (\hat{u}_h - u_h^+) n^+, \\ \hat{q}_h^- = q_h^- + \tau_{qu}^- (\hat{u}_h - u_h^-) n^- + \tau_{qp}^- (\hat{p}_h^- - p_h^-) n^-, \end{cases}$$

and

$$(2.2e) \quad \llbracket \hat{q}_h \rrbracket(x_i) = 0, \quad \llbracket \hat{p}_h \rrbracket(x_i) = 0 \quad \text{for } i = 1, \dots, N-1,$$

$$(2.2f) \quad \hat{u}_h = u_D \text{ on } \partial\Omega, \quad \hat{q}_h = q_N \text{ on } \partial\Omega_N.$$

To define the initial approximation (u_h^0, q_h^0, p_h^0) for the semi-discrete scheme above, we use the HDG method for stationary third-order equations in [8] to solve the equation $v + v_{xxx} = g$, where $g = u_0 + (u_0)_{xxx}$ with u_0 being the initial data of the time-dependent problem (1.1). It is easy to see that the approximate HDG solution to v is a good approximation to u_0 , and the approximate solutions to v_x and v_{xx} satisfy (2.2a)-(2.2b). Therefore, we take the HDG approximation to (v, v_x, v_{xx}) as the initial approximation (u_h^0, q_h^0, p_h^0) for the semi-discrete scheme (2.2). This way of choosing initial data for time-dependent problems by solving corresponding stationary problems has been used in [12, 9].

It is not difficult to define HDG methods that are associated to other characterizations of the exact solution, but these methods are actually the same, provided that the corresponding stabilization function allows for the transition from one characterization to the other; see [16, 8]. In fact, the choice of characterization to use is more relevant for the actual implementation of the HDG method rather than for its actual definition.

Next, we present our stability result and a priori error analysis of the HDG method for the linear problem under some conditions on the stabilization function.

2.3.1. *Stability.* For the HDG method defined by (2.2), we first have the following stability result.

Theorem 2.1. *If the stabilization function in the numerical traces (2.2d) satisfies*

$$(2.3) \quad -\tau_{pu}^+ - \frac{1}{2}\tau_{qu}^+{}^2 \geq 0 \quad \text{and} \quad \tau_{qu}^-\tau_{qp}^- - \frac{1}{2} \geq 0,$$

and at least one of the two inequalities is strict, then the HDG method (2.2) is stable and has a unique solution.

2.3.2. *A priori error estimate.* Now we consider the convergence properties of our HDG method (2.2). We proceed as follows. We first define an auxiliary projection and state its optimal approximation property. Then, we provide an estimate for the L^2 -norm of the projections of the errors in the primary and auxiliary variables.

Let us introduce a key auxiliary projection that is tailored to the numerical traces. The projection of the function $(u, q, p) \in H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h)$, $\Pi(u, q, p) := (\Pi u, \Pi q, \Pi p)$, is defined as follows. On an element $I_i = (x_{i-1}, x_i)$, the projection is the element of $P_k(I_i) \times P_k(I_i) \times P_k(I_i)$ which solves the following equations:

$$(2.4a) \quad (\delta_u, v)_{I_i} = 0 \quad \forall v \in P_{k-1}(I_i),$$

$$(2.4b) \quad (\delta_q, z)_{I_i} = 0 \quad \forall z \in P_{k-1}(I_i),$$

$$(2.4c) \quad (\delta_p, w)_{I_i} = 0 \quad \forall w \in P_{k-1}(I_i),$$

$$(2.4d) \quad \delta_p - \tau_{pu}^+ \delta_u n = 0 \quad \text{on } x_{i-1}^+,$$

$$(2.4e) \quad \delta_q - \tau_{qu}^+ \delta_u n = 0 \quad \text{on } x_{i-1}^+,$$

$$(2.4f) \quad \delta_q - \tau_{qu}^- \delta_u n - \tau_{qp}^- \delta_p n = 0 \quad \text{on } x_i^-,$$

where we use the notation $\delta_\omega := \omega - \Pi\omega$ for $\omega = u, q$, and p . Note that the last three equations have exactly the same structure as the numerical traces of the HDG method in (2.2d).

The following result for the optimal approximation properties of the projection Π was shown in [8]. To state it, we use the following notation. The $H^s(D)$ -norm is denoted by $\|\cdot\|_{s,D}$. We drop the first subindex if $s = 0$, and the second one if $D = \Omega$ or $D = \mathcal{T}_h$.

Lemma 2.2. *Suppose that*

$$(2.5) \quad \tau_{qu}^+ + \tau_{qu}^- - \tau_{pu}^+ \tau_{qp}^- \neq 0.$$

Then the projection Π in (2.4) is well defined on any interval I_i . In addition, if $\tau_{qu}^+, \tau_{qu}^-, \tau_{pu}^+$ and τ_{qp}^- are constants, we have that, for $\omega = u, q$ and p , there is a constant C such that

$$\|\omega - \Pi\omega\|_{I_i} \leq C h^{s+1} \quad \text{for } s \in [1, k],$$

provided $\omega \in H^{s+1}(I_i)$.

Next, we provide estimates for the L^2 -norm of the projection of the errors

$$\epsilon_u := \Pi u - u_h, \quad \epsilon_q := \Pi q - q_h, \quad \epsilon_p := \Pi p - p_h,$$

and deduce from them the estimates for the L^2 -norm of the errors

$$e_u := u - u_h, \quad e_q := q - q_h, \quad e_p := p - p_h.$$

Theorem 2.3. Suppose that $F(u) = 0$ in the problem (2.1) and the exact solution $(u, q, p) \in W^{2,\infty}((0, T]; H^{k+1}(\mathcal{T}_h)) \times W^{1,\infty}((0, T]; H^{k+1}(\mathcal{T}_h)) \times W^{1,\infty}((0, T]; H^{k+1}(\mathcal{T}_h))$. If the stabilization function of the HDG method (2.2) satisfies the condition

$$(2.6) \quad \begin{aligned} &\tau_{qu}^- > 0, \quad \tau_{qu}^- \tau_{qp}^- = 1, \quad \text{and} \\ &\tau_{qu}^+ \in [0, 1], \quad \tau_{pu}^+ \in [-1 - \sqrt{1 - \tau_{qu}^{+2}}, -\frac{1}{2} - \frac{1}{2}\tau_{qu}^{+2}], \end{aligned}$$

then for $k > 0$ and h small enough, we have

$$\|\epsilon_u(t)\| + \|\epsilon_q(t)\| + \|\epsilon_p(t)\| + \|\epsilon_{u_t}(t)\| \leq Ch^{k+1} \quad \text{for } 0 \leq t \leq T,$$

where C is independent of h .

It is easy to see that if the stabilization function satisfies the condition (2.6), then it also satisfies the conditions (2.3) and (2.5). Using Lemma 2.2, Theorem 2.3 and the triangle inequality, we immediately get the following L^2 error estimate for the actual errors.

Theorem 2.4. Suppose that the hypotheses of Theorem 2.3 are satisfied. Then we have

$$\|e_u(t)\| + \|e_q(t)\| + \|e_p(t)\| + \|e_{u_t}(t)\| \leq Ch^{k+1} \quad \text{for } 0 \leq t \leq T,$$

where C is independent of h .

2.4. HDG method for the nonlinear problem. In this section, we consider the problem (2.1) with the nonlinear term $F(u) \neq 0$. The approximation provided by the HDG method, $(u_h, q_h, p_h, \hat{u}_h, \hat{q}_h, \hat{p}_h)$, is the element of $W_h^k \times W_h^k \times W_h^k \times L^2(\mathcal{E}_h) \times L^2(\partial\mathcal{T}_h) \times L^2(\partial\mathcal{T}_h)$ which solves the equations

$$\begin{aligned} (2.7a) \quad &(q_h, v)_{\mathcal{T}_h} + (u_h, v_x)_{\mathcal{T}_h} - \langle \hat{u}_h, vn \rangle_{\partial\mathcal{T}_h} = 0, \\ (2.7b) \quad &(p_h, z)_{\mathcal{T}_h} + (q_h, z_x)_{\mathcal{T}_h} - \langle \hat{q}_h, zn \rangle_{\partial\mathcal{T}_h} = 0, \\ (2.7c) \quad &(u_{h_t}, w)_{\mathcal{T}_h} - (p_h + F(u_h), w_x)_{\mathcal{T}_h} + \langle \hat{p}_h + \hat{F}_h, wn \rangle_{\partial\mathcal{T}_h} = (f, w)_{\mathcal{T}_h}, \end{aligned}$$

for all $(v, z, w) \in W_h^k \times W_h^k \times W_h^k$, where, on $\partial\mathcal{T}_h$, we have

$$(2.7d) \quad \begin{cases} \hat{p}_h^+ = p_h^+ + \tau_{pu}^+ (\hat{u}_h - u_h^+) n^+, \\ \hat{q}_h^+ = q_h^+ + \tau_{qu}^+ (\hat{u}_h - u_h^+) n^+, \\ \hat{q}_h^- = q_h^- + \tau_{qu}^- (\hat{u}_h - u_h^-) n^- + \tau_{qp}^- (\hat{p}_h^- - p_h^-) n^-, \\ \hat{F}_h = F(\hat{u}_h) - \tau_F(\hat{u}_h, u_h) (\hat{u}_h - u_h) n, \end{cases}$$

and

$$(2.7e) \quad \llbracket \hat{q}_h \rrbracket(x_i) = 0, \quad \llbracket \hat{p}_h + \hat{F}_h \rrbracket(x_i) = 0 \quad \text{for } i = 1, \dots, N-1,$$

$$(2.7f) \quad \hat{u}_h = u_D \text{ on } \partial\Omega, \quad \hat{q}_h = q_N \text{ on } \partial\Omega_N.$$

Note that due to the nonlinearity of F , the stabilization function $\tau_F(\cdot, \cdot) : \partial\mathcal{T}_h \rightarrow \mathbb{R}$ in (2.7d) can be a nonlinear function of \hat{u}_h and u_h . Let

$$\tilde{\tau}(u_h, \hat{u}_h) := \frac{1}{(u_h - \hat{u}_h)^2} \int_{\hat{u}_h}^{u_h} (F(s) - F(\hat{u}_h)) n ds.$$

We have the following stability result.

Theorem 2.5. Assume that $u_D = q_N = 0$. If the stabilization function satisfies

$$(2.8) \quad \begin{aligned} &(\tau_F^+ - \tilde{\tau}^+) - \tau_{pu}^+ - \frac{1}{2}(\tau_{qu}^+)^2 \geq 0, \text{ and} \\ &(\tau_F^- - \tilde{\tau}^-) + \frac{1}{2}(\tau_{qu}^-)^2 \geq 0, \quad (\tau_F^- - \tilde{\tau}^-)(\tau_{qp}^-)^2 + \tau_{qu}^- \tau_{qp}^- - \frac{1}{2} \geq 0, \end{aligned}$$

then for the method (2.7) we have

$$\frac{d}{dt} \|u_h\|^2 \leq (f, u_h).$$

Note that if $F(u) = 0$, then we can take $\tau_F = \tilde{\tau} = 0$ and the condition (2.8) becomes the same as the condition (2.3). If $F(u) \neq 0$, we just need to have $\tau_F \geq \tilde{\tau}$ and take $\tau_{qu}^\pm, \tau_{pu}^+$ and τ_{qp}^- to satisfy (2.3) as in the linear case. Since

$$\tilde{\tau} = \frac{1}{(u_h - \hat{u}_h)^2} \int_{\hat{u}_h}^{u_h} F'(\xi)(s - \hat{u}_h)n \, ds \leq \frac{1}{2} \sup_{s \in J(u_h, \hat{u}_h)} |F'(s)|,$$

where $J(u_h, \hat{u}_h) = [\min\{u_h, \hat{u}_h\}, \max\{u_h, \hat{u}_h\}]$, the stabilization function τ_F satisfies the condition $\tau_F \geq \tilde{\tau}$ if

$$\tau_F \geq \frac{1}{2} \sup_{s \in J(u_h, \hat{u}_h)} |F'(s)|.$$

For other choices of τ_F which satisfies the condition $\tau_F \geq \tilde{\tau}$, see [23].

3. PROOFS

In this section, we provide detailed proofs of our main results. We first prove Theorem 2.1 and Theorem 2.3, which are the L^2 -stability and error estimates of the method for linear equations. Then we prove the L^2 -stability of the method in the nonlinear case in Theorem 2.5.

In the following analysis, we will use the notation

$$(\varphi, v) = (\varphi, v)_{\mathcal{T}_h} \quad \text{and} \quad \langle \varphi, vn \rangle = \langle \varphi, vn \rangle_{\partial \mathcal{T}_h}.$$

3.1. L^2 -Stability in the linear case. We start by proving Theorem 2.1 using an energy argument.

Proof. To show that the method is L^2 -stable and has a unique solution, we only need to prove that the solution of (2.2) is trivial if $f = 0$, $u_0 = 0$, and the boundary data $u_D = q_N = 0$.

Taking $v = -p_h$, $z = q_h$ and $\omega = u_h$ in (2.2a)–(2.2c) and adding the three equations together, we get

$$\begin{aligned} 0 &= -(q_h, p_h) - (u_h, p_{hx}) + \langle \hat{u}_h, p_h n \rangle \\ &\quad + (p_h, q_h) + (q_h, q_{hx}) - \langle \hat{q}_h, q_h n \rangle \\ &\quad + (u_{ht}, u_h) - (p_h, u_{hx}) + \langle \hat{p}_h, u_h n \rangle \\ &= \frac{1}{2} \frac{d}{dt} \|u_h\|^2 + \langle \hat{p}_h u_h + \hat{u}_h p_h - p_h u_h, n \rangle + \langle \frac{1}{2} q_h^2 - \hat{q}_h q_h, n \rangle. \end{aligned}$$

by using integration by parts. Note that \hat{u}_h , \hat{q}_h and \hat{p}_h are single-valued, $\hat{u}_h = 0$ on $\partial\Omega$, and $\hat{q}_h = 0$ on $\partial\Omega_N$ by (2.2e) and (2.2f). We have

$$0 = \frac{1}{2} \frac{d}{dt} \|u_h\|^2 - \langle \hat{p}_h - p_h, (\hat{u}_h - u_h)n \rangle + \frac{1}{2} \langle (\hat{q}_h - q_h)^2, n \rangle + \frac{1}{2} \hat{q}_h^2(x_0).$$

With the obvious notation, the equation above is written as

$$(3.1) \quad 0 = \frac{1}{2} \frac{d}{dt} \|u_h\|^2 + \frac{1}{2} \hat{q}_h^2(x_0) + A^+ + A^-,$$

where

$$\begin{aligned} A^+ &= -\langle \hat{p}_h - p_h, (\hat{u}_h - u_h)n \rangle_{\partial\mathcal{T}_h^+} + \frac{1}{2} \langle (\hat{q}_h - q_h)^2, n \rangle_{\partial\mathcal{T}_h^+}, \\ A^- &= -\langle \hat{p}_h - p_h, (\hat{u}_h - u_h)n \rangle_{\partial\mathcal{T}_h^-} + \frac{1}{2} \langle (\hat{q}_h - q_h)^2, n \rangle_{\partial\mathcal{T}_h^-}. \end{aligned}$$

Using the definition of the numerical traces in (2.2d), we get

$$\begin{aligned} A^+ &= -\langle \tau_{pu}^+, (\hat{u}_h - u_h)^2 \rangle_{\partial\mathcal{T}_h^+} + \frac{1}{2} \langle (\tau_{qu}^+)^2, (\hat{u}_h - u_h)^2 \rangle_{\partial\mathcal{T}_h^+} \\ &= \langle -\tau_{pu}^+ - \frac{1}{2} (\tau_{qu}^+)^2, (\hat{u}_h - u_h)^2 \rangle_{\partial\mathcal{T}_h^+}, \end{aligned}$$

and

$$\begin{aligned} A^- &= -\langle \hat{p}_h - p_h, (\hat{u}_h - u_h)n \rangle_{\partial\mathcal{T}_h^-} + \frac{1}{2} \langle 1, (\tau_{qu}^- (\hat{u}_h - u_h)n + \tau_{qp}^- (\hat{p}_h - p_h)n)^2 \rangle_{\partial\mathcal{T}_h^-} \\ &= +\frac{1}{2} \langle (\tau_{qu}^-)^2, (\hat{u}_h - u_h)^2 \rangle_{\partial\mathcal{T}_h^-} + \frac{1}{2} \langle (\tau_{qp}^-)^2, (\hat{p}_h - p_h)^2 \rangle_{\partial\mathcal{T}_h^-} \\ &\quad + \langle \tau_{qu}^- \tau_{qp}^- - 1, (\hat{u}_h - u_h)(\hat{p}_h - p_h) \rangle_{\partial\mathcal{T}_h^-}. \end{aligned}$$

If the stabilization function satisfies the condition (2.3), then we get $A^+ \geq 0$ and $A^- \geq 0$, and the equation (3.1) implies that

$$\frac{d}{dt} \|u_h\|^2 \leq 0.$$

Since $u_h = 0$ for $t = 0$, we have $u_h = 0$ for any $t > 0$. Therefore, from (3.1) we obtain that $A^+ = 0$ and $A^- = 0$. Since at least one of the two inequalities in the condition (2.3) is strict, we have either $\hat{u}_h = u_h^+$ or $\hat{u}_h = u_h^-$, which implies that $\hat{u}_h = 0$. So we can easily get $q_h = 0$ from (2.2a) and then $\hat{q}_h = 0$ from (2.2d)–(2.2f). From (2.2b), we obtain that $p_h = 0$ which implies $\hat{p}_h = 0$. This completes the proof. \square

3.2. Error analysis in the linear case. In this section, we prove the optimal error estimate for the projections of the errors in Theorem 2.3. First, we obtain the equations for the projection of the errors.

3.2.1. The error equations. From the equations defining the HDG method, (2.2a)–(2.2c), and the fact that the exact solution also satisfy these equations, we obtain the following error equations

$$\begin{aligned} (e_q, v) + (e_u, v_x) - \langle \hat{e}_u, vn \rangle &= 0, \\ (e_p, z) + (e_q, z_x) - \langle \hat{e}_q, zn \rangle &= 0, \\ (e_{ut}, w) - (e_p, w_x) + \langle \hat{e}_p, wn \rangle &= 0, \end{aligned}$$

for all $v, z, w \in W_h^k \times W_h^k \times W_h^k$, where $\hat{e}_\omega = \omega - \hat{\omega}_h$ for $\omega = u, q$, and p . From (2.2d)–(2.2f), it is easy to see that

$$\begin{cases} \hat{e}_p^+ = e_p^+ + \tau_{pu}^+ (\hat{e}_u - e_u^+) n^+, \\ \hat{e}_q^+ = e_q^+ + \tau_{qu}^+ (\hat{e}_u - e_u^+) n^+, \\ \hat{e}_q^- = e_q^- + \tau_{qu}^- (\hat{e}_u - e_u^-) n^- - \tau_{qp}^- (\hat{e}_p^- - e_p^-) n^-, \end{cases}$$

and

$$\begin{aligned} \llbracket \widehat{e}_q \rrbracket(x_i) &= 0, \quad \llbracket \widehat{e}_p \rrbracket(x_i) = 0 \quad \text{for } i = 1, \dots, N-1, \\ \widehat{e}_u &= 0 \text{ on } \partial\Omega, \quad \widehat{e}_q = 0 \text{ on } \partial\Omega_N. \end{aligned}$$

Now we set

$$\widehat{\epsilon}_u = \widehat{e}_u \quad \text{and} \quad \widehat{\epsilon}_p^- = \widehat{e}_p^-,$$

and let

$$(3.2) \quad \begin{cases} \widehat{\epsilon}_p^+ = \epsilon_p^+ + \tau_{pu}^+ (\widehat{\epsilon}_u - \epsilon_u^+) n^+, \\ \widehat{\epsilon}_q^+ = \epsilon_q^+ + \tau_{qu}^+ (\widehat{\epsilon}_u - \epsilon_u^+) n^+, \\ \widehat{\epsilon}_q^- = \epsilon_q^- + \tau_{qu}^- (\widehat{\epsilon}_u - \epsilon_u^-) n^- + \tau_{qp}^- (\widehat{\epsilon}_p^- - \epsilon_p^-) n^-. \end{cases}$$

Using the definition of the projection Π , (2.4d)–(2.4f), we see that

$$\widehat{\epsilon}_p^+ = \widehat{e}_p^+ \quad \text{and} \quad \widehat{\epsilon}_q^\pm = \widehat{e}_q^\pm.$$

Therefore, by the definition of the projection Π , (2.4a)–(2.4c), we easily get the following equations for the projections of errors

$$(3.3a) \quad (\epsilon_q, v) + (\delta_q, v) + (\epsilon_u, v_x) - \langle \widehat{\epsilon}_u, vn \rangle = 0,$$

$$(3.3b) \quad (\epsilon_p, z) + (\delta_p, z) + (\epsilon_q, z_x) - \langle \widehat{\epsilon}_q, zn \rangle = 0,$$

$$(3.3c) \quad (\epsilon_{ut}, w) + (\delta_{ut}, w) - (\epsilon_p, w_x) + \langle \widehat{\epsilon}_p, wn \rangle = 0,$$

for all $v, z, w \in W_h^k \times W_h^k \times W_h^k$, and

$$(3.3d) \quad \llbracket \widehat{\epsilon}_q \rrbracket(x_i) = 0, \quad \llbracket \widehat{\epsilon}_p \rrbracket(x_i) = 0 \quad \text{for } i = 1, \dots, N-1,$$

$$(3.3e) \quad \widehat{\epsilon}_u = 0 \text{ on } \partial\Omega, \quad \widehat{\epsilon}_q = 0 \text{ on } \partial\Omega_N.$$

3.2.2. Energy identities. To prove the L^2 -error estimate in Theorem 2.3, we begin by establishing a key identity involving the quantity

$$\|\epsilon\|^2 := \|\epsilon_u\|^2 + \|\epsilon_q\|^2 + \|\epsilon_p\|^2 + \|\epsilon_{ut}\|^2.$$

Lemma 3.1. *We have that*

$$\frac{1}{2} \frac{d}{dt} \|\epsilon\|^2 + S + \Psi = 0,$$

where

$$\begin{aligned} S &= (\delta_{ut}, \epsilon_u) - (\delta_q, \epsilon_p) + (\delta_p, \epsilon_q) + (\delta_{qt}, \epsilon_q) + (\delta_p, \epsilon_{ut}) - (\delta_{ut}, \epsilon_p) \\ &\quad + (\delta_{pt}, \epsilon_p) - (\delta_{ut}, \epsilon_{qt}) + (\delta_{qt}, \epsilon_{ut}) + (\delta_{utt}, \epsilon_{ut}) - (\delta_{qt}, \epsilon_{pt}) + (\delta_{pt}, \epsilon_{qt}), \\ \Psi &= -\langle \widehat{\epsilon}_p - \epsilon_p, (\widehat{\epsilon}_u - \epsilon_u) n \rangle + \frac{1}{2} \langle (\widehat{\epsilon}_q - \epsilon_q)^2, n \rangle \\ &\quad + \langle \widehat{\epsilon}_q - \epsilon_q, (\widehat{\epsilon}_{ut} - \epsilon_{ut}) n \rangle + \frac{1}{2} \langle (\widehat{\epsilon}_p - \epsilon_p)^2, n \rangle \\ &\quad + \langle \widehat{\epsilon}_{qt} - \epsilon_{qt}, (\widehat{\epsilon}_p - \epsilon_p) n \rangle + \frac{1}{2} \langle (\widehat{\epsilon}_{ut} - \epsilon_{ut})^2, n \rangle \\ &\quad - \langle \widehat{\epsilon}_{pt} - \epsilon_{pt}, (\widehat{\epsilon}_{ut} - \epsilon_{ut}) n \rangle + \frac{1}{2} \langle (\widehat{\epsilon}_{qt} - \epsilon_{qt})^2, n \rangle \\ &\quad + \frac{1}{2} \widehat{\epsilon}_q^2(x_0) + \frac{1}{2} (\widehat{\epsilon}_p + \widehat{\epsilon}_{qt})^2(x_0) - \frac{1}{2} \widehat{\epsilon}_p^2(x_N). \end{aligned}$$

Proof. Differentiating the error equations (3.3a)–(3.3c) with respect to t , we get

$$(3.4a) \quad (\epsilon_{qt}, v) + (\delta_{qt}, v) + (\epsilon_{ut}, v_x) - \langle \widehat{\epsilon}_{ut}, vn \rangle = 0,$$

$$(3.4b) \quad (\epsilon_{pt}, z) + (\delta_{pt}, z) + (\epsilon_{qt}, z_x) - \langle \widehat{\epsilon}_{qt}, zn \rangle = 0,$$

$$(3.4c) \quad (\epsilon_{utt}, w) + (\delta_{utt}, w) - (\epsilon_{pt}, w_x) + \langle \widehat{\epsilon}_{pt}, wn \rangle = 0,$$

Next, we use (3.3) and (3.4) to get four energy identities.

(i) Taking $w = \epsilon_u$, $v = -\epsilon_p$, and $z = \epsilon_q$ in (3.3) and adding the three equations together, we have

$$\begin{aligned} 0 = & (\epsilon_{ut}, \epsilon_u) + (\delta_{ut}, \epsilon_u) - (\epsilon_p, \epsilon_{ux}) + \langle \widehat{\epsilon}_p, \epsilon_u n \rangle \\ & - (\epsilon_q, \epsilon_p) - (\delta_q, \epsilon_p) - (\epsilon_u, \epsilon_{px}) + \langle \widehat{\epsilon}_u, \epsilon_p n \rangle \\ & + (\epsilon_p, \epsilon_q) + (\delta_p, \epsilon_q) + (\epsilon_q, \epsilon_{qx}) - \langle \widehat{\epsilon}_q, \epsilon_q n \rangle. \end{aligned}$$

Using integration by parts and (3.3d) and (3.3e), we get

$$(3.5) \quad \begin{aligned} 0 = & \frac{1}{2} \frac{d}{dt} \|\epsilon_u\|^2 + (\delta_{ut}, \epsilon_u) - (\delta_q, \epsilon_p) + (\delta_p, \epsilon_q) \\ & - \langle \widehat{\epsilon}_p - \epsilon_p, (\widehat{\epsilon}_u - \epsilon_u) n \rangle + \frac{1}{2} \langle (\widehat{\epsilon}_q - \epsilon_q)^2, n \rangle + \frac{1}{2} \widehat{\epsilon}_q^2(x_0). \end{aligned}$$

(ii) Similar to (i), taking $v = \epsilon_q$ in (3.4a), $z = \epsilon_{ut}$ in (3.3b), and $w = -\epsilon_p$ in (3.3c) and adding the equations together, we get

$$(3.6) \quad \begin{aligned} 0 = & \frac{1}{2} \frac{d}{dt} \|\epsilon_q\|^2 + (\delta_{qt}, \epsilon_q) + (\delta_p, \epsilon_{ut}) - (\delta_{ut}, \epsilon_p) \\ & + \langle \widehat{\epsilon}_q - \epsilon_q, (\widehat{\epsilon}_{ut} - \epsilon_{ut}) n \rangle + \frac{1}{2} \langle (\widehat{\epsilon}_p - \epsilon_p)^2, n \rangle - \frac{1}{2} \widehat{\epsilon}_p^2(x_N) + \frac{1}{2} \widehat{\epsilon}_p^2(x_0). \end{aligned}$$

(iii) Taking $v = \epsilon_{ut}$ in (3.4a), $z = \epsilon_p$ in (3.4b), and $w = -\epsilon_{qt}$ in (3.3c) and adding the equations together, we get

$$(3.7) \quad \begin{aligned} 0 = & \frac{1}{2} \frac{d}{dt} \|\epsilon_p\|^2 + (\delta_{pt}, \epsilon_p) + (\delta_{qt}, \epsilon_{ut}) - (\delta_{ut}, \epsilon_{qt}) \\ & + \langle \widehat{\epsilon}_{qt} - \epsilon_{qt}, (\widehat{\epsilon}_p - \epsilon_p) n \rangle + \frac{1}{2} \langle (\widehat{\epsilon}_{ut} - \epsilon_{ut})^2, n \rangle + \widehat{\epsilon}_{qt} \widehat{\epsilon}_p(x_0). \end{aligned}$$

(iv) Taking $v = -\epsilon_{pt}$, $z = \epsilon_{qt}$, and $w = \epsilon_{ut}$ in (3.4a)–(3.4c), respectively, and adding the equations together, we get

$$(3.8) \quad \begin{aligned} 0 = & \frac{1}{2} \frac{d}{dt} \|\epsilon_{ut}\|^2 + (\delta_{utt}, \epsilon_{ut}) - (\delta_{qt}, \epsilon_{pt}) + (\delta_{pt}, \epsilon_{qt}) \\ & - \langle \widehat{\epsilon}_{pt} - \epsilon_{pt}, (\widehat{\epsilon}_{ut} - \epsilon_{ut}) n \rangle + \frac{1}{2} \langle (\widehat{\epsilon}_{qt} - \epsilon_{qt})^2, n \rangle + \frac{1}{2} \widehat{\epsilon}_{qt}^2(x_0). \end{aligned}$$

The proof is completed by adding the four equations (3.5)–(3.8) together. \square

3.2.3. Proof of the L^2 -error estimates. Using Lemma 3.1, we first get the following result.

Lemma 3.2. *If the stabilization function satisfies the condition (2.6), then we have*

$$\|\epsilon(t)\|^2 \leq \|\epsilon(0)\|^2 + \Theta(0) + \int_0^t \widehat{\epsilon}_p^2(x_N) dt + 2 \left| \int_0^t S dt \right| \quad \text{for } 0 \leq t \leq T,$$

where

$$\Theta = \langle \tau_{qu}^+ - \tau_{pu}^+ \tau_{qu}^+, (\widehat{\epsilon}_u - \epsilon_u)^2 \rangle_{\partial \mathcal{T}_h^+} + \langle 1, \tau_{qu}^- (\widehat{\epsilon}_u - \epsilon_u)^2 + \tau_{qp}^- (\widehat{\epsilon}_p - \epsilon_p)^2 \rangle_{\partial \mathcal{T}_h^-},$$

and S is the same as in Lemma 3.1.

Proof. Using the definition of $\widehat{\epsilon}_p^+$ and $\widehat{\epsilon}_q$, (3.2), for the Ψ term in Lemma 3.1, we have

$$\Psi = \Psi^+ + \Psi^-,$$

where

$$\begin{aligned} \Psi^+ = & -\langle \tau_{pu}^+, (\widehat{\epsilon}_u - \epsilon_u)^2 \rangle_{\partial \mathcal{T}_h^+} - \frac{1}{2} \langle (\tau_{qu}^+)^2, (\widehat{\epsilon}_u - \epsilon_u)^2 \rangle_{\partial \mathcal{T}_h^+} \\ & + \langle \tau_{qu}^+, (\widehat{\epsilon}_u - \epsilon_u)(\widehat{\epsilon}_u - \epsilon_u)_t \rangle_{\partial \mathcal{T}_h^+} - \frac{1}{2} \langle (\tau_{pu}^+)^2, (\widehat{\epsilon}_u - \epsilon_u)^2 \rangle_{\partial \mathcal{T}_h^+} \\ & - \langle \tau_{pu}^+ \tau_{qu}^+, (\widehat{\epsilon}_u - \epsilon_u)(\widehat{\epsilon}_u - \epsilon_u)_t \rangle_{\partial \mathcal{T}_h^+} - \frac{1}{2} \langle 1, (\widehat{\epsilon}_{ut} - \epsilon_{ut})^2 \rangle_{\partial \mathcal{T}_h^+} \\ & - \langle \tau_{pu}^+, (\widehat{\epsilon}_{ut} - \epsilon_{ut})^2 \rangle_{\partial \mathcal{T}_h^+} - \frac{1}{2} \langle (\tau_{qu}^+)^2, (\widehat{\epsilon}_{ut} - \epsilon_{ut})^2 \rangle_{\partial \mathcal{T}_h^+} \\ & + \frac{1}{2} \widehat{\epsilon}_q^2(x_0) + \frac{1}{2} (\widehat{\epsilon}_p + \widehat{\epsilon}_{qt})^2(x_0) \end{aligned}$$

and

$$\begin{aligned} \Psi^- = & -\langle \widehat{\epsilon}_p - \epsilon_p, \widehat{\epsilon}_u - \epsilon_u \rangle_{\partial \mathcal{T}_h^-} + \frac{1}{2} \langle 1, (\tau_{qu}^- (\widehat{\epsilon}_u - \epsilon_u) + \tau_{qp}^- (\widehat{\epsilon}_p - \epsilon_p))^2 \rangle_{\partial \mathcal{T}_h^-} \\ & + \langle \tau_{qu}^- (\widehat{\epsilon}_u - \epsilon_u) + \tau_{qp}^- (\widehat{\epsilon}_p - \epsilon_p), (\widehat{\epsilon}_u - \epsilon_u)_t \rangle_{\partial \mathcal{T}_h^-} + \frac{1}{2} \langle 1, (\widehat{\epsilon}_p - \epsilon_p)^2 \rangle_{\partial \mathcal{T}_h^-} \\ & + \langle \tau_{qu}^- (\widehat{\epsilon}_u - \epsilon_u)_t + \tau_{qp}^- (\widehat{\epsilon}_p - \epsilon_p)_t, \widehat{\epsilon}_p - \epsilon_p \rangle_{\partial \mathcal{T}_h^-} + \frac{1}{2} \langle 1, (\widehat{\epsilon}_{ut} - \epsilon_{ut})^2 \rangle_{\partial \mathcal{T}_h^-} \\ & - \langle \widehat{\epsilon}_{pt} - \epsilon_{pt}, \widehat{\epsilon}_{ut} - \epsilon_{ut} \rangle_{\partial \mathcal{T}_h^-} + \frac{1}{2} \langle 1, (\tau_{qu}^- (\widehat{\epsilon}_{ut} - \epsilon_{ut}) + \tau_{qp}^- (\widehat{\epsilon}_{pt} - \epsilon_{pt}))^2 \rangle_{\partial \mathcal{T}_h^-} \\ & - \frac{1}{2} \widehat{\epsilon}_p^2(x_N). \end{aligned}$$

We can rewrite the term Ψ^+ as

$$\Psi^+ = \Gamma_1 + \frac{1}{2} \frac{d}{dt} \Theta_1,$$

where

$$\begin{aligned} \Gamma_1 = & \langle -\tau_{pu}^+ - \frac{1}{2} (\tau_{qu}^+)^2 - \frac{1}{2} (\tau_{pu}^+)^2, (\widehat{\epsilon}_u - \epsilon_u)^2 \rangle_{\partial \mathcal{T}_h^+} \\ & + \langle -\frac{1}{2} - \tau_{pu}^+ - \frac{1}{2} (\tau_{qu}^+)^2, (\widehat{\epsilon}_{ut} - \epsilon_{ut})^2 \rangle_{\partial \mathcal{T}_h^+} + \frac{1}{2} \widehat{\epsilon}_q^2(x_0) + \frac{1}{2} (\widehat{\epsilon}_p + \widehat{\epsilon}_{qt})^2(x_0), \\ \Theta_1 = & \langle \tau_{qu}^+ - \tau_{pu}^+ \tau_{qu}^+, (\widehat{\epsilon}_u - \epsilon_u)^2 \rangle_{\partial \mathcal{T}_h^+}. \end{aligned}$$

Similarly, if we assume that $\tau_{qu}^- \tau_{qp}^- = 1$, after some calculations we get

$$\Psi^- = \Gamma_2 + \frac{1}{2} \frac{d}{dt} \Theta_2 - \frac{1}{2} \widehat{\epsilon}_p^2(x_N),$$

where

$$\begin{aligned} \Gamma_2 = & \langle (\frac{1}{2} \tau_{qu}^-)^2, (\widehat{\epsilon}_u - \epsilon_u)^2 \rangle_{\partial \mathcal{T}_h^-} + \langle \frac{1}{2}, \left(\tau_{qp}^- (\widehat{\epsilon}_p - \epsilon_p) + (\widehat{\epsilon}_u - \epsilon_u)_t \right)^2 \rangle_{\partial \mathcal{T}_h^-} \\ & + \langle (\frac{1}{2} \tau_{qp}^-)^2, (\widehat{\epsilon}_{pt} - \epsilon_{pt})^2 \rangle_{\partial \mathcal{T}_h^-} + \langle \frac{1}{2}, \left((\widehat{\epsilon}_p - \epsilon_p) + \tau_{qu}^- (\widehat{\epsilon}_u - \epsilon_u)_t \right)^2 \rangle_{\partial \mathcal{T}_h^-}, \\ \Theta_2 = & \langle 1, \tau_{qu}^- (\widehat{\epsilon}_u - \epsilon_u)^2 + \tau_{qp}^- (\widehat{\epsilon}_p - \epsilon_p)^2 \rangle_{\partial \mathcal{T}_h^-}. \end{aligned}$$

So from Lemma 3.1 we get

$$(3.9) \quad \frac{1}{2} \frac{d}{dt} (\|\epsilon\|^2 + \Theta_1 + \Theta_2) + \Gamma_1 + \Gamma_2 = \frac{1}{2} \hat{\epsilon}_p^2(x_N) - S.$$

Now we integrate the equation (3.9) with respect to t and get

$$\begin{aligned} & \frac{1}{2} \left(\|\epsilon(t)\|^2 + \Theta_1(t) + \Theta_2(t) \right) + \int_0^t (\Gamma_1 + \Gamma_2) dt \\ &= \frac{1}{2} \left(\|\epsilon(0)\|^2 + \Theta_1(0) + \Theta_2(0) \right) + \frac{1}{2} \int_0^t \hat{\epsilon}_p^2(x_N) dt - \int_0^t S dt. \end{aligned}$$

It is easy to check that if $\tau_{qu}^\pm, \tau_{pu}^+$ and τ_{qp}^- satisfy the condition (2.6), we have

$$\Theta_1 \geq 0, \quad \Theta_2 \geq 0, \quad \Gamma_1 \geq 0, \quad \Gamma_2 \geq 0 \quad \text{for any } t \in [0, T].$$

Therefore,

$$\|\epsilon(t)\|^2 \leq \|\epsilon(0)\|^2 + \Theta(0) + \int_0^t \hat{\epsilon}_p^2(x_N) dt + 2 \left| \int_0^t S dt \right|,$$

where $\Theta = \Theta_1 + \Theta_2$. □

To prove Theorem 2.3, we also need the following error estimates for the initial approximations at $t = 0$ (See Theorem 2.2 and Theorem 2.3 in [8]).

Lemma 3.3. *If $\tau_{qu}^\pm, \tau_{pu}^+, \tau_{qp}^-$ satisfy the condition (2.5), then for $k > 0$,*

$$\begin{aligned} \|\epsilon_u(0)\| + \|\epsilon_q(0)\| + \|\epsilon_p(0)\| &\leq Ch^{k+2}, \\ \|\hat{\epsilon}_u(0)\|_{\mathcal{E}_h} + \|\hat{\epsilon}_q(0)\|_{\mathcal{E}_h} + \|\hat{\epsilon}_p(0)\|_{\mathcal{E}_h} &\leq Ch^{2k+1}. \end{aligned}$$

In addition, let us get an estimate for ϵ_{ut} at $t = 0$.

Lemma 3.4. *If $\tau_{qu}^\pm, \tau_{pu}^+, \tau_{qp}^-$ satisfy the condition (2.5), then for $k > 0$*

$$\|\epsilon_{ut}(0)\| \leq Ch^{k+1}.$$

Proof. Taking $t = 0$ and $w = \epsilon_{ut}(0)$ in the error equation (3.3c), we have

$$(\epsilon_{ut}(0), \epsilon_{ut}(0)) + (\delta_{ut}(0), \epsilon_{ut}(0)) - (\epsilon_p(0), \epsilon_{ut}(0)) + \langle \hat{\epsilon}_p(0), \epsilon_{ut}(0)n \rangle = 0.$$

By Cauchy inequality, trace inequality and inverse inequality, we get

$$\|\epsilon_{ut}(0)\|^2 \leq C \|\delta_{ut}(0)\|^2 + Ch^{-2} \|\epsilon_p(0)\|^2 + Ch^{-1} \|\hat{\epsilon}_p(0)\|_{\mathcal{E}_h}^2.$$

Then the conclusion follows by using Lemma 2.2 and Lemma 3.3. □

Now let us finish the proof of Theorem 2.3 by estimating the right hand side of the inequality in Lemma 3.2 and using Lemma 3.3 and Lemma 3.4.

Proof. We first estimate the term $\int_0^t \hat{\epsilon}_p^2(x_N) dt$. Taking ω to be $\omega_1 := \frac{x-x_0}{x_N-x_0}$ in (3.3c), we get

$$\hat{\epsilon}_p(x_N) = -(\epsilon_{ut}, \omega_1) - (\delta_{ut}, \omega_1) + \left(\epsilon_p, \frac{1}{x_N - x_0} \right),$$

since $\omega_1(x_0) = 0$ and $\omega_1(x_N) = 1$. Using Cauchy inequality, we have

$$\begin{aligned} |\hat{\epsilon}_p(x_N)| &\leq |(\epsilon_{ut}, \omega_1)| + |(\delta_{ut}, \omega_1)| + \left| \left(\epsilon_p, \frac{1}{x_N - x_0} \right) \right| \\ &\leq C(\|\epsilon_{ut}\| + \|\delta_{ut}\| + \|\epsilon_p\|). \end{aligned}$$

Then by the approximation property of the projection Π in Lemma 2.2, we obtain

$$(3.10) \quad \int_0^t \widehat{\epsilon}_p^2(x_N) dt \leq Ch^{2k+2} + \int_0^t \|\epsilon\|^2 dt.$$

Next, we estimate the term $|\int_0^t S dt|$. Let

$$S = S_1 + S_2,$$

where

$$\begin{aligned} S_1 &= (\delta_{ut}, \epsilon_u) - (\delta_q, \epsilon_p) + (\delta_p, \epsilon_q) + (\delta_{qt}, \epsilon_q) + (\delta_p, \epsilon_{ut}) - (\delta_{ut}, \epsilon_p) \\ &\quad + (\delta_{pt}, \epsilon_p) + (\delta_{qt}, \epsilon_{ut}) + (\delta_{utt}, \epsilon_{ut}), \\ S_2 &= -(\delta_{ut}, \epsilon_{qt}) - (\delta_{qt}, \epsilon_{pt}) + (\delta_{pt}, \epsilon_{qt}). \end{aligned}$$

Using Cauchy inequality and the approximation property of the projection Π in Lemma (2.2), we get

$$\int_0^t |S_1| dt \leq Ch^{k+1} \int_0^t \|\epsilon\| dt.$$

Integrating S_2 with respect to t , we have

$$\begin{aligned} \int_0^t S_2 dt &= -(\delta_{ut}, \epsilon_q)|_0^t + \int_0^t (\delta_{utt}, \epsilon_q) dt - (\delta_{qt}, \epsilon_p)|_0^t + \int_0^t (\delta_{qtt}, \epsilon_p) dt \\ &\quad + (\delta_{pt}, \epsilon_q)|_0^t - \int_0^t (\delta_{ptt}, \epsilon_q) dt. \end{aligned}$$

By the approximation property of the projection Π in Lemma 2.2,

$$\left| \int_0^t S_2 dt \right| \leq Ch^{2k+2} + C\|\epsilon(0)\|^2 + \frac{1}{4}\|\epsilon(t)\|^2 + Ch^{k+1} \int_0^t \|\epsilon\| dt.$$

So we get

$$\begin{aligned} (3.11) \quad \left| \int_0^t S dt \right| &\leq \int_0^t |S_1| dt + \left| \int_0^t S_2 dt \right| \\ &\leq Ch^{2k+2} + C\|\epsilon(0)\|^2 + \frac{1}{4}\|\epsilon(t)\|^2 + Ch^{k+1} \int_0^t \|\epsilon\| dt. \end{aligned}$$

Applying (3.10) and (3.11) to Lemma 3.2, we have

$$\|\epsilon(t)\|^2 \leq C\|\epsilon(0)\|^2 + C\Theta(0) + Ch^{2k+2} + C \int_0^t \|\epsilon(t)\|^2 dt.$$

Since

$$\Theta(0) \leq C(\|\widehat{\epsilon}_u(0)\|_{\mathcal{E}_h}^2 + \|\widehat{\epsilon}_p(0)\|_{\mathcal{E}_h}^2) + Ch^{-1}(\|\epsilon_u(0)\|^2 + \|\epsilon_p(0)\|^2)$$

by Lemma 3.3 and the trace inequality, we have

$$\|\epsilon(t)\|^2 \leq Ch^{2k+2} + C \int_0^t \|\epsilon(t)\|^2 dt$$

using Lemma 3.3 and Lemma 3.4. Now we use Grönwall's inequality and get

$$\|\epsilon(t)\|^2 \leq Ch^{2k+2},$$

where C depends on t but not on h . This completes the proof of Theorem 2.3. \square

3.3. L^2 -stability in the nonlinear case. Now let us prove Theorem 2.5 on the stability of the HDG method for the nonlinear KdV equation. We treat the nonlinear term in a similar way to that in [23].

Proof. Taking $\omega = u_h, v = -p_h$ and $z = q_h$ in (2.7a)–(2.7c) and adding the three equations together, we get

$$\begin{aligned} (f, u_h) = & (u_{ht}, u_h) - (p_h + F(u_h), u_{hx}) + \langle \hat{p}_h + \hat{F}_h, u_h n \rangle \\ & - (q_h, p_h) - (u_h, p_{hx}) + \langle \hat{u}_h, p_h n \rangle \\ & + (p_h, q_h) + (q_h, q_{hx}) - \langle \hat{q}_h, q_h n \rangle. \end{aligned}$$

Using integration by parts and (2.7e)–(2.7f), we have

$$\begin{aligned} (3.12) \quad (f, u_h) = & \frac{1}{2} \frac{d}{dt} \|u_h\|^2 - (F(u_h), u_{hx}) - \langle \hat{p}_h + \hat{F}_h - p_h, (\hat{u}_h - u_h) n \rangle \\ & + \frac{1}{2} \langle (\hat{q}_h - q_h)^2, n \rangle + \frac{1}{2} \hat{q}_h^2(0). \end{aligned}$$

Let $G(s)$ be such that $dG(s)/ds = F(s)$. It is easy to see that

$$-(F(u_h), u_{hx}) = -\left(\frac{d}{dx} G(u_h), 1\right) = -\langle G(u_h), n \rangle = -\left\langle \int_{\hat{u}_h}^{u_h} F(s) ds, n \right\rangle.$$

Using it for the second term in (3.12), we get that

$$(f, u_h) = \frac{1}{2} \frac{d}{dt} \|u_h\|^2 + \Phi + \frac{1}{2} \hat{q}_h^2(0),$$

where

$$\begin{aligned} \Phi = & -\left\langle \int_{\hat{u}_h}^{u_h} (F(s) - F(\hat{u}_h)) ds, n \right\rangle - \langle \hat{F}_h - F(\hat{u}_h), (\hat{u}_h - u_h) n \rangle \\ & - \langle \hat{p}_h - p_h, (\hat{u}_h - u_h) n \rangle + \frac{1}{2} \langle (\hat{q}_h - q_h)^2, n \rangle. \end{aligned}$$

Let $\tilde{\tau} := \frac{1}{(\hat{u}_h - u_h)^2} \int_{\hat{u}_h}^{u_h} (F(s) - F(\hat{u}_h)) n ds$. Using the definition of \hat{F}_h in (2.7d), we have

$$\Phi = \langle \tau_F - \tilde{\tau}, (\hat{u}_h - u_h)^2 \rangle - \langle \hat{p}_h - p_h, (\hat{u}_h - u_h) n \rangle + \frac{1}{2} \langle (\hat{q}_h - q_h)^2, n \rangle.$$

By the definition of \hat{p}_h in (2.7d), we get

$$\begin{aligned} \Phi^+ := \Phi|_{\mathcal{T}_h^+} = & \langle \tau_F^+ - \tilde{\tau}^+ - \tau_{pu}^+ - \frac{1}{2} (\tau_{qu}^+)^2, (\hat{u}_h - u_h)^2 \rangle_{\partial \mathcal{T}_h^+}, \\ \Phi^- := \Phi|_{\mathcal{T}_h^-} = & \langle \tau_F^- - \tilde{\tau}^- + \frac{1}{2} (\tau_{qu}^-)^2, (\hat{u}_h - u_h)^2 \rangle_{\partial \mathcal{T}_h^-} + \left\langle \frac{1}{2} (\tau_{qp}^-)^2, (\hat{p}_h - p_h)^2 \right\rangle_{\partial \mathcal{T}_h^-} \\ & + \langle \tau_{qu}^- \tau_{qp}^- - 1, (\hat{p}_h - p_h)(\hat{u}_h - u_h) n \rangle_{\partial \mathcal{T}_h^-}. \end{aligned}$$

It is easy to check that if the stabilization function satisfies the condition (2.8), then we get $\Phi^+ \geq 0$ and $\Phi^- \geq 0$. This shows that

$$\frac{d}{dt} \|u_h\|^2 \leq (f, u_h).$$

□

4. NUMERICAL RESULTS

In this section, we carry out several numerical experiments to study the accuracy and capability of our HDG method. In the first and the second numerical experiments, we examine the orders of convergence of the method for linear and nonlinear third-order problems. In the third and the fourth experiments, we apply the method to solve some well-known dispersive wave problems. For all the experiments, we use the following second-order midpoint rule [5, 9] for time discretization. Let $0 = t_0 < t_1 < \dots < t_J = T$ be a partition of the interval $[0, T]$ and $\Delta t_j = t_{j+1} - t_j$. For $j = 0, \dots, J-1$ and $\omega \in \{u_h, q_h, p_h\}$, let $\omega^{j+1} \in W_h^k$ be defined as

$$\omega^{j+1} = 2\omega^{j,1} - \omega^j,$$

where $\omega^{j,1}$ is the solution of the equation

$$\frac{\omega^{j,1} - \omega^j}{\frac{1}{2}\Delta t_j} + (\omega^{j,1})_{xxx} + F(\omega^{j,1})_x = 0.$$

The components of the stabilization function, $(\tau_{qu}^+, \tau_{pu}^+, \tau_{qu}^-, \tau_{qp}^-)$ are taken to be $(0, -1, 1, 1)$ in all the following numerical tests.

Numerical experiment 1: In this test, we use the HDG method to solve the time-dependent third-order linear problem

$$u_t + u_{xxx} = f,$$

where f is chosen so that the exact solution is $u(x, t) = \sin(x + t)$ on the domain $(x, t) \in [0, 1] \times [0, 0.1]$. The initial condition is $u_0 = \sin(x)$ and the boundary conditions are $u(0, t) = \sin(t)$, $u(1, t) = \sin(1 + t)$ and $u_x(1, t) = \cos(1 + t)$. We take $h = 2^{-n}$ for $n = 1, \dots, 5$, $\Delta t = 0.1 * h^2$ for $k = 0, 1$, and $\Delta t = 0.1 * h^3$ for $k = 2, 3$ to compute the orders of convergence of u_h, q_h, p_h at the final time $T = 0.1$. These orders we observe in the numerical experiments are listed in Table 1.

Our numerical results indicate that the orders of convergence of (e_u, e_q, e_p) are optimal as predicted by the error estimate in Theorem 2.4 for any $k > 0$. For $k = 0$, although our error analysis is inclusive, we observe that the method converges optimally in the numerical experiment.

Numerical experiment 2: Now we use the HDG method to solve the nonlinear third-order equation

$$u_t + u_{xxx} + (3u^2)_x = f,$$

where f is chosen so that the exact solution is $u(x, t) = \sin(2x + t)$ in the domain $(x, t) \in [0, \pi] \times [0, 0.1]$. The initial condition and the boundary conditions are extracted from the exact solution and included in the initial-boundary-value problem (IVBP). Here, we take the stabilization function $\tau_F = 3$, given that $F(u) = 3u^2$ and $\frac{1}{2}|F'(u)| = 3|u| \leq 3$ for the solution u . The mesh size for the HDG method is $h = 2^{-n}$ for $n = 3, \dots, 7$. The step size for time discretization is $\Delta t = 0.1 * h^2$ for $k = 0, 1$ and $\Delta t = 0.1 * h^3$ for $k = 2, 3$. The orders of convergence of u_h, q_h, p_h at the final time $T = 0.1$ are displayed in Table 2. Our numerical results show that the orders of convergence of (e_u, e_q, e_p) are also optimal for any $k \geq 0$ for the nonlinear problem.

In the previous two tests, we have observed optimal convergence rates of the HDG method for both linear and nonlinear third-order problems. In the next two tests, we apply the methods to solve the KdV equation

$$(4.1) \quad u_t + u_{xxx} + (3u^2)_x = 0.$$

k	e_u	Order	e_q	Order	e_p	Order
0	1.27e-01	-	1.07e-01	-	1.94e-01	-
	6.87e-02	0.89	6.26e-02	0.77	1.13e-01	0.78
	3.83e-02	0.84	3.52e-02	0.83	6.10e-02	0.89
	2.08e-02	0.88	1.94e-02	0.86	3.31e-02	0.88
	1.07e-02	0.96	1.03e-02	0.92	1.85e-02	0.84
1	1.13e-02	-	1.22e-02	-	6.83e-03	-
	3.28e-03	1.79	3.08e-03	1.99	1.90e-03	1.85
	8.62e-04	1.93	7.69e-04	2.00	4.87e-04	1.97
	2.17e-04	1.99	1.92e-04	2.00	1.22e-04	1.99
	5.44e-05	2.00	4.80e-05	2.00	3.06e-05	2.00
2	3.66e-04	-	3.27e-04	-	7.41e-04	-
	4.59e-05	2.99	4.33e-05	2.92	6.99e-05	3.41
	5.71e-06	3.01	5.50e-06	2.98	1.12e-05	2.64
	7.10e-07	3.01	6.94e-07	2.99	1.49e-06	2.91
	8.86e-08	3.00	8.73e-08	2.99	1.90e-07	2.97
3	1.97e-05	-	5.43e-05	-	7.32e-04	-
	1.05e-06	4.23	2.24e-06	4.60	8.53e-05	3.10
	6.50e-08	4.01	7.77e-08	4.85	4.19e-06	4.35
	4.07e-09	4.00	3.88e-09	4.32	1.86e-07	4.49
	2.55e-10	4.00	2.32e-10	4.06	5.68e-09	5.03

TABLE 1. The error (e_u, e_q, e_p) and their convergence orders for the linear problem in the numerical experiment 1.

Numerical experiment 3: Here, we consider the classical *solitary-wave solution* [5, 26]

$$u(x, t) = 2 \operatorname{sech}^2(x - 4t + 4)$$

of the KdV equation (4.1) in the domain $(x, t) \in [-10, 0] \times [0, 2]$. The initial condition and boundary conditions are extracted from the exact solution to form the IBVP.

In the computation, we use 100 elements, piecewise cubic polynomials, and time-step size $\Delta t = 10^{-3}$, and take $\tau_F = (F'(\hat{u}))^2 + \frac{1}{4}$ so that $\tau_F > \frac{1}{2}|F'(\hat{u}_h)|$. The space-time graphs of the computed solution (u_h, q_h, p_h) as well as the exact solutions (u, q, p) at the final time $T = 2$ are displayed in Figure 1. We observe a good match between the approximate solutions and the exact solutions.

Numerical experiment 4: In this test, we simulate the interaction of two solitary waves with different propagation speeds using our HDG method. The KdV equation (4.1) with the initial condition

$$u_0(x) = 5 \frac{4.5 \operatorname{csch}^2[1.5(x + 14.5)] + 2 \operatorname{sech}^2(x + 12)}{\{3 \coth[1.5(x + 14.5)] - 2 \tanh(x + 12)\}^2}$$

k	e_u	Order	e_q	Order	e_p	Order
0	6.63e-01	-	1.34e-00	-	2.63e-00	-
	4.08e-01	0.70	8.58e-01	0.64	1.79e-00	0.56
	2.37e-01	0.78	5.17e-01	0.73	1.16e-00	0.64
	1.32e-01	0.84	2.94e-01	0.82	6.78e-01	0.76
	7.11e-02	0.90	1.59e-01	0.89	3.71e-01	0.87
1	5.35e-02	-	9.60e-02	-	2.31e-01	-
	1.29e-02	2.05	2.36e-02	2.03	5.29e-02	2.12
	3.18e-03	2.02	5.86e-03	2.01	1.28e-02	2.05
	7.92e-04	2.01	1.47e-03	2.00	3.17e-03	2.01
	1.98e-04	2.00	3.67e-04	2.00	7.92e-04	2.00
2	3.31e-03	-	5.81e-03	-	1.25e-02	-
	4.01e-04	3.05	7.32e-04	2.99	1.61e-03	2.96
	4.97e-05	3.01	9.20e-05	2.99	1.99e-04	3.01
	6.20e-06	3.00	1.15e-05	3.00	2.48e-05	3.00
	7.74e-07	3.00	1.44e-06	3.00	3.10e-06	3.00
3	1.54e-04	-	2.81e-04	-	6.52e-04	-
	9.57e-06	4.01	1.77e-05	3.99	3.82e-05	4.09
	5.97e-07	4.00	1.17e-06	3.99	2.39e-06	4.00
	3.73e-08	4.00	6.97e-08	4.00	1.49e-07	4.00
	2.33e-09	4.00	4.36e-09	4.00	1.03e-08	3.86

TABLE 2. The error (e_u, e_q, e_p) and their convergence orders for the nonlinear problem in the numerical experiment 2.

has the solution

$$u(x, t) = 5 \frac{4.5 \operatorname{csch}^2[1.5(x - 9t + 14.5)] + 2 \operatorname{sech}^2(x - 4t + 12)}{\{3 \coth[1.5(x - 9t + 14.5)] - 2 \tanh(x - 4t + 12)\}^2}$$

(see [26]). We choose the domain $(x, t) \in [-20, 0] \times [0, 2]$ and the boundary data $u(-20, t)$, $u(0, t)$, $u_x(0, t)$ are extracted from the exact solution.

In our computation, we use 50 elements, piecewise cubic polynomials, and the time-step size $\Delta t = 10^{-4}$. The stabilization function τ_F is taken in the same way as in the previous test. The space-time graphs of the HDG approximate solutions and the exact solutions are displayed in Figure 2. From the side-by-side comparison, we see that the HDG solutions are good approximations to the exact solutions. They show that the two waves are moving toward the same direction. The faster soliton catches up with the slower one and they overlap around $t = 0.5$. Afterwards, the faster soliton continues to propagate and the slower one falls behind.

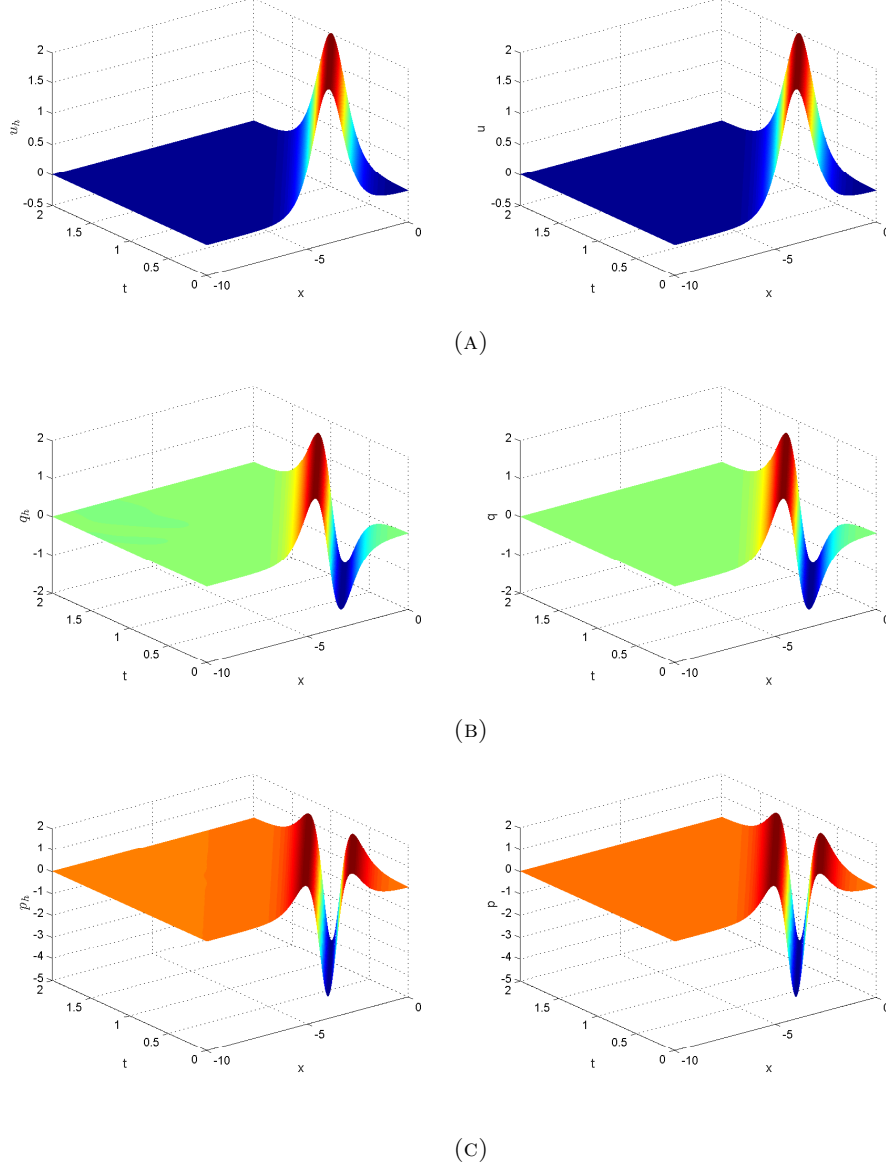
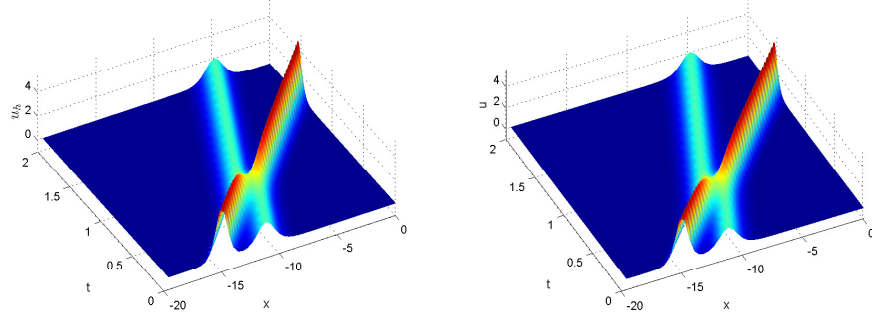


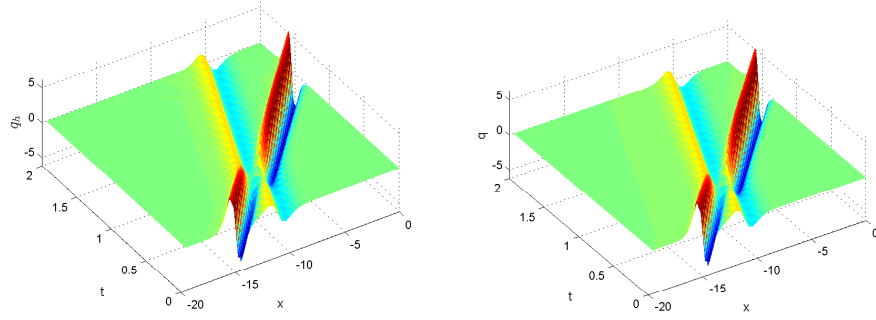
FIGURE 1. Space-time graphs of one soliton in the domain $(x, t) \in [-10, 0] \times [0, 2]$. Evolution of the HDG approximate solution (left) and the exact solution (right) of (A): u , (B): q , and (C): p .

5. CONCLUDING REMARKS

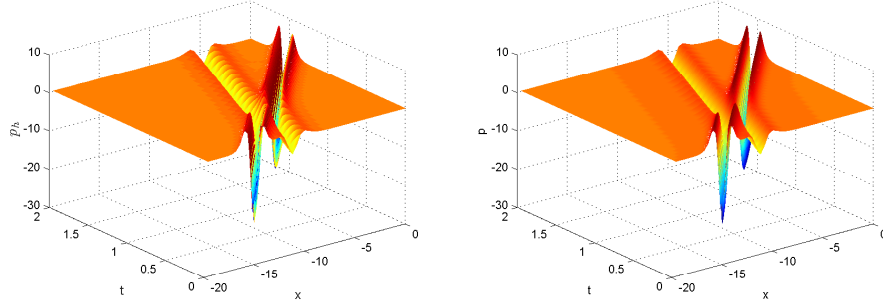
In this paper, we develop a new HDG method for time-dependent third-order equations in one space dimension based on the characterization of the exact solution as the solutions to local problems that are “glue” together by transmission conditions. We find conditions on the stabilization function under which the method



(A)



(B)



(C)

FIGURE 2. Space-time graphs of the interaction of two solitary waves in the domain $(x, t) \in [-20, 0] \times [0, 2]$. Evolution of the HDG approximate solution (left) and the exact solution (right) of (A): u , (B): q , and (C): p .

is L^2 stable for generalized KdV type equations. We also prove optimal error estimates for the linear third-order equation. Numerical results from computation

verify the theoretical error analysis and show that the methods are able to accurately simulate solitary wave solutions of the KdV equation. Our future work is to develop and analyze HDG methods for third-order equations in multiple dimensions and complex systems.

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